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Explicit modular map for abelian surfaces via $K3$ surfaces

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1 Introduction

1.1 Purpose and Method

We use the following notations:

$\mathcal{X}_A$: the total family of principally polarized (in short p.p.) abelian surfaces,

$\mathcal{K}$: the total family of algebraic Kummer surfaces,

$\mathcal{X}_{A_5}$: the total family of p.p. abelian surfaces with real multiplication by $\boldsymbol{Q}(\sqrt{5})$,

$\mathcal{K}_5$: the total family of Kummer surfaces corresponding to $\mathcal{X}_{A_5}$.

The purpose of this article is to show an explicit description of the modular map for $\mathcal{X}_A$ and $\mathcal{X}_{A_5}$. Here, "explicit" means

(i) an exact defining equation of the surfaces with parameters fitting with the compactification of the moduli space,

(ii) an exact system of modular functions defined on the period domain that makes possible our approximate calculations,

(iii) an exact definition of the period map such that its inverse map coincides with (ii)

(iv) a description of the period differential equation (if possible).

For this purpose, we use some kind of families of $K3$ surfaces those are equivalent to $\mathcal{X}_A$ or $\mathcal{X}_{A_5}$ as deformation families of the complex structure.

1.2 Two period domains

Suppose an abelian surface A.

(1) The usual period matrix is given by ${}^t\begin{pmatrix}\Omega_1\\ \Omega_2\end{pmatrix} = \left(\int_{\gamma_i}\omega_k\right)$, $(1\leq i\leq 4, k=1,2)$ of the periods of the holomorphic 1- forms ω_k along 1-cycles γ_i. Here the system $\{\gamma_1,\ldots,\gamma_4\}$ is a symplectic basis with $\gamma_i\cdot\gamma_{i+2}=-1$ $(i=1,2)$. The normalized period matrix is given by $\Omega=\Omega_1\Omega_2^{-1}$. It belongs to the Siegel space $\mathfrak{S}_2$.

(2) The holomorphic 2-form is given by $\varphi=\omega_1\wedge\omega_2$. By taking six $(2,2)$ minors of the extended normalized period matrix ${}^t\begin{pmatrix}g & h & 1 & 0\\ h & g' & 0 & 1\end{pmatrix}$ we get a map

$$\omega:\begin{pmatrix}g & h\\ h & g'\end{pmatrix}\mapsto(\eta_1:\cdots:\eta_5) = \left(\int_{\gamma_2\wedge\gamma_4}\varphi,\int_{\gamma_1\wedge\gamma_4}\varphi,\int_{-\gamma_2\wedge\gamma_3}\varphi,\int_{\gamma_1\wedge\gamma_2}\varphi,\int_{\gamma_3\wedge\gamma_4}\varphi\right)$$
$$=(h:g:g':h^2-gg':1).$$

It holds

$$\mathrm{Im}\begin{pmatrix} g & h \\ h & g' \end{pmatrix} > 0 \Longleftrightarrow \mathrm{Im}\ \eta_2 > 0, \eta B_0 {}^t\overline{\eta} > 0.$$

Here, we have $-\int_{\gamma_2\wedge\gamma_4}\varphi = \int_{\gamma_1\wedge\gamma_3}\varphi$. It means $C_c = \gamma_1\wedge\gamma_3 + \gamma_2\wedge\gamma_4$ is an algebraic cycle. By taking the orthogonal compelement $C_c^{\perp}$ in $H_2(A, \boldsymbol{Z})$, we obtain

$$B_0 = \langle -2\rangle \oplus U \oplus U, U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence, we have equivalent period domains $\mathfrak{S}_2 \cong$

$$\mathscr{D}_{B_0} = \{\eta = (\eta_1 : \ldots : \eta_5) \in \boldsymbol{P}^4 : \eta B_0 {}^t\eta = 0, \eta B_0 {}^t\overline{\eta} > 0, \eta_2 > 0\}.$$

The map ω induces an explicit isomorphism

$$\omega^* : PO^+(B_0, \boldsymbol{Z}) \xrightarrow{\sim} Sp(4, \boldsymbol{Z}).$$

1.3 $K3$ surfaces

Let S be a $K3$ surface (note that a Kummer surface is a $K3$ surface). Always it holds

$$H_2(S, \boldsymbol{Z}) = L_{K3} = E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U,\ U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\mathrm{NS}(S)$ be a sublattice in L_{K3} which is generated by divisors on S. Its signature is always $(1, *)$. Let $\mathrm{Tr}(S)$ be its orthogonal complement. Its signature is always $(2, *)$.

Let $\mathcal{F}_0$ be a family of $K3$ surfaces which generic member S has $\mathrm{Tr}(S) \cong B_0$, namely $\mathrm{NS}(S) \cong L_0 = E_8(-1) \oplus E_7(-1) \oplus U$, with a fixed marking. Here we omit the exact definition of this marking (for detail see [N-S]).

The Torelli type theorem: We have a bijective correspondence by the period map between the family $\mathcal{F}_0$ of isomorphism classes of marked $K3$ surfaces and the period domain $\mathscr{D}_{B_0}$.

Hence, $\mathcal{X}_A$ and $\mathcal{F}_0$ have the common period domain $\mathscr{D}_{B_0}$. It is the same for the family $\mathcal{K}$.

2 Family of elliptic $K3$ surfaces

2.1 Clingher-Doran's $\mathcal{F}_{CD}$

According to Clingher-Doran [C-D], we take a family of elliptic $K3$ surfaces with complex parameters $\alpha, \beta, \gamma, \delta$:

$$S = S(\alpha, \beta, \gamma, \delta) : y^2 = x^3 + (-3\alpha t^4 - \gamma t^5)x + (t^5 - 2\beta t^6 + \delta t^7). \tag{2.1}$$

The fibration is given by $\pi : (t, x, y) \mapsto t$. For a generic member, it holds $\mathrm{NS}(S) = E_8(-1) \oplus E_7(-1) \oplus U$ with singular fibers of type II^*, of type III^* and five of type I_1's, we say in short with the singular composition $II^* + III^* + 5I_1$. In Fig. 2.1, s_0 indicates the holomorphic section given by the points at infinity at every fiber, and f indicates a generic fiber.

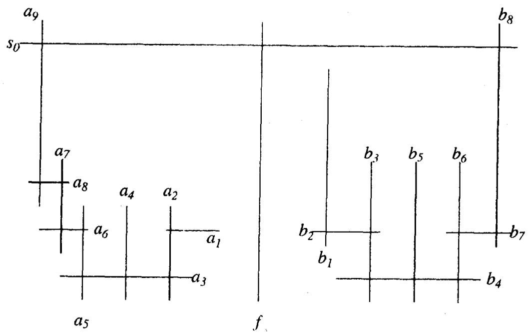

Fig. 2.1: Singular fibers of S

We set $\mathcal{F}_{CD} := \{S(\alpha,\beta,\gamma,\delta)\}$. The following is our key property.

Theorem 2.1. *It holds $\mathcal{F}_0 = \mathcal{F}_{CD}$.*

To show it, we must start from the exact definitions of $\mathcal{F}_0$ and $\mathcal{F}_{CD}$ as families of marked $K3$ surfaces. Stll more, we need a detailed argument about the isomorphism of marked pairings. We omit them (for detail see [N-S]).

2.2 Kummer surfaces

By a change of the fibration $t = \dfrac{1}{2z_1}, x = \dfrac{s}{2z_1^2}$, we have another expression equipped with a 2-torsion section $z_1 = y_1 = 0$

$$\begin{cases} X_{CD} = X_{CD}(\alpha,\beta,\gamma,\delta) : y_1^2 = z_1^3 + \mathcal{P}_X(s)z_1^2 + \mathcal{Q}_X(s)z_1 \\ \mathcal{P}_X(s) = 4s^3 - 3\alpha s - \beta, \\ \mathcal{Q}_X(s) = \frac{1}{4}(\delta - 2\gamma s). \end{cases} \tag{2.2}$$

By the fiberwise two isogeny map

$$(z_1, y_1) \mapsto (x,y) = \left(\frac{y_1^2}{z_1^2}, \frac{(\mathcal{Q}_X(s) - z_1^2)y_1}{z_1^2}\right),$$

we obtain its quotient manifold

$$\begin{cases} Y_{CD} = Y_{CD}(\alpha,\beta,\gamma,\delta) : y^2 = z^3 + \mathcal{P}_Y(s)z^2 + \mathcal{Q}_Y(s)z \\ \mathcal{P}_Y(s) = -2\mathcal{P}_X(s) = -8s^3 + 6\alpha s + 2\beta, \\ \mathcal{Q}_Y(s) = \mathcal{P}_X(s)^2 - 4\mathcal{Q}_X(s) \\ \quad = 16s^6 - 24\alpha s^4 - 8\beta s^3 + 9\alpha^2 s^2 + 2(3\alpha\beta + \gamma)s + (\beta^2 - \delta). \end{cases} \tag{2.3}$$

It becomes to be a Kummer surface with the same period as $S(\alpha,\beta,\gamma,\delta)$ (for detail see [N-S]).

2.3 A Shimura variety

According to Theorem 2.1, for a member $S = S(\alpha,\beta,\gamma,\delta) \in \mathcal{F}_{CD}$, we may identify a p.p. abelian surface $A(S)$ that has the same period with that of S. If $A(S)$ has a real multiplication by $\sqrt{5}$, we say that S has the same property.

Theorem 2.2. *The surface $S(\alpha,\beta,\gamma,\delta) \in \mathcal{F}_0$ has a real multiplication by $\sqrt{5}$ if and only if*

$$(-\alpha^3 - \beta^2 + \delta)^2 - 4\alpha(\alpha\beta - \gamma)^2 = 0. \tag{2.4}$$

Remark 2.1. *$S(\alpha,\beta,\gamma,\delta)$ is a rational elliptic surface for $\gamma=\delta=0$. Otherwise it is an elliptic $K3$ surface. Degenerating locus (i.e. the singular composition is not generic) is given by*

$$\begin{aligned}C_r:\ &\gamma(6\alpha\beta\gamma+\gamma^2+9\alpha^2\delta)(-23328\alpha^6\beta\gamma^3+46656\alpha^3\beta^3\gamma^3-23328\beta^5\gamma^3-3888\alpha^5\gamma^4\\&+97200\alpha^2\beta^2\gamma^4+33750\alpha\beta\gamma^5+3125\gamma^6-34992\alpha^7\gamma^2\delta+69984\alpha^4\beta^2\gamma^2\delta-34992\alpha\beta^4\gamma^2\delta\\&+184680\alpha^3\beta\gamma^3\delta+48600\beta^3\gamma^3\delta+37125\alpha^2\gamma^4\delta+71928\alpha^4\gamma^2\delta^2+68040\alpha\beta^2\gamma^2\delta^2-27000\beta\gamma^3\delta^2\\&+11664\alpha^6\delta^3-23328\alpha^3\beta^2\delta^3+11664\beta^4\delta^3-46656\alpha^2\beta\gamma\delta^3-48600\alpha\gamma^2\delta^3\\&-23328\alpha^3\delta^4-23328\beta^2\delta^4+11664\delta^5)=0.\end{aligned}$$

Remark 2.2. *(a) We are considering the family of isomorphism classes of marked $K3$ surfaces (with some special marking) by $\mathcal{F}_0$.*

(b) That is the family of isomorphism classes of (some special) elliptic $K3$ surfaces.

(c) $S(\alpha,\beta,\gamma,\delta)$ and $S(\alpha',\beta',\gamma',\delta')$ are isomorphic (as elliptic surfaces) if and only if they lie on the same orbit of two $\boldsymbol{C}^*$ actions $(x,y)\mapsto(x',y')=(k^2x,k^3y), t\mapsto t'=mt$.

(d) It means that we get the weighted projective space $\boldsymbol{P}(2,3,5,6)$ as the compactification of the space of parameters $(\alpha,\beta,\gamma,\delta)$.

(e) Via the period map and theTorelli theorem, we know that the compactified moduli space is $\boldsymbol{P}(2,3,5,6)$. According to Remark 2.1, this is the compactification by attaching $\boldsymbol{P}^1=\{(\alpha,\beta,\gamma,\delta)\in\boldsymbol{P}(2,3,5,6):\gamma=\delta=0\}$.

2.4 Nagano's family $\mathcal{F}_N$ for $\sqrt{5}$

According to A. Nagano [N3], we have

Theorem 2.3. *(1) The family of Kummer surfaces with $\sqrt{5}$ action is given by*

$$Z_N(\mathcal{A},\mathcal{B},\mathcal{C}):w^2=(u^2-2y^5)(u-(5\mathcal{A}y^2-10\mathcal{B}y+\mathcal{C})),\tag{2.5}$$

with $(\mathcal{A},\mathcal{B},\mathcal{C})\in\boldsymbol{P}(1,3,5)-\{(1,0,0)\}$.
(2) Let $\mathcal{F}_N$ be the family of $Z_N(\mathcal{A},\mathcal{B},\mathcal{C})$. The parameters $\mathcal{A},\mathcal{B},\mathcal{C}$ are described by some symmetric Hilbert modular forms of weight $2,6,10$, resp..
(3) The period map is constructed geometrically. It gives a biholomorphic correspondence between the compactified parameter space $\boldsymbol{P}(1,3,5)$ and the one point compactification of the period domain $\boldsymbol{H}\times\boldsymbol{H}/\langle SL(2,\mathcal{O}_k),\iota\rangle$, where ι is the involution of the coordinates exchange of $\boldsymbol{H}\times\boldsymbol{H}$.

Theorem 2.4. *We have the equivalence of the deformation families (under some markings):*

$$\mathcal{F}_N\cong\mathcal{K}_5.\tag{2.6}$$

3 Quartic Kummer surface

3.1 Rosenhein's formula

Start from a curve of genus 2:

$$\begin{cases}C(\lambda)=C(\lambda_1,\lambda_2,\lambda_3):y^2=x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3),\\(\lambda_1,\lambda_2,\lambda_3)\in\Lambda=\{(\lambda_1,\lambda_2,\lambda_3)\in\boldsymbol{C}^3:\lambda_i\notin\{0,1\},\lambda_i\neq\lambda_j\}\end{cases}$$

Set $\lambda_0=(-0.6,-0.3,0.6)$, $C_0=C(\lambda_0)$, and set a symplectic basis $\{A_1,A_2,B_1,B_2\}$ of $H_1(C_0,\boldsymbol{Z})$ as in Fig. 3.1.

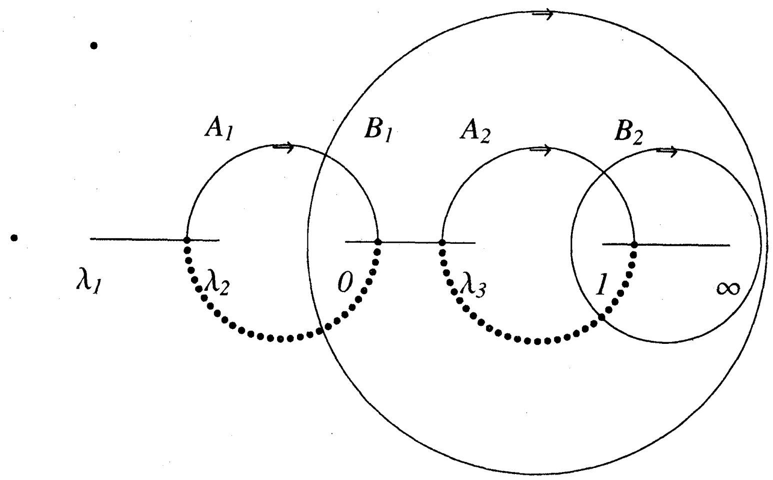

Fig.3.1: Homology cycles of C_0

For $\{\omega_1 = \frac{dx}{y}, \omega_2 = \frac{xfx}{y}\}$, the period of C_0 is given by

$$\begin{pmatrix}\Omega_1\\ \Omega_2\end{pmatrix} = {}^t\begin{pmatrix}\int_{B_1}\omega_1 & \int_{B_2}\omega_1 & \int_{A_1}\omega_1 & \int_{A_2}\omega_1\\ \int_{B_1}\omega_2 & \int_{B_2}\omega_2 & \int_{A_1}\omega_2 & \int_{A_2}\omega_2\end{pmatrix}$$

$$\Omega = \Omega_1\Omega_2^{-1} \in \mathfrak{S}_2.$$

We extend this procedure in a small neighborhood U_0 of λ_0 , and we get a local period map $\lambda \mapsto \Omega, \lambda \in U_0$. By the analytic continuation we define the global period map $\Phi_C : \Lambda \to \mathfrak{S}_2$. Under this setting we can define Riemann theta constants

$$\vartheta\begin{bmatrix}a_1 & a_2\\ b_1 & b_2\end{bmatrix} = \vartheta\begin{bmatrix}a_1 & a_2\\ b_1 & b_2\end{bmatrix}(\Omega) = \sum_{n\in \boldsymbol{Z}^2}\exp[\pi i(a/2+n)\Omega^t(a/2+n) + 2\pi i(a/2+n)^t b/2]$$

with $a = (a_1, a_2), b = (b_1, b_2) \in \{0,1\}^2$.

Theorem 3.1. (Classical Rosenhein type formula) *The inverse of the period map is given by*

$$\lambda_1 = -\frac{\vartheta^2\begin{bmatrix}1&1\\1&1\end{bmatrix}\vartheta^2\begin{bmatrix}0&0\\0&1\end{bmatrix}}{\vartheta^2\begin{bmatrix}0&1\\1&0\end{bmatrix}\vartheta^2\begin{bmatrix}1&0\\0&0\end{bmatrix}}, \lambda_2 = -\frac{\vartheta^2\begin{bmatrix}1&1\\1&1\end{bmatrix}\vartheta^2\begin{bmatrix}1&0\\0&1\end{bmatrix}}{\vartheta^2\begin{bmatrix}0&1\\1&0\end{bmatrix}\vartheta^2\begin{bmatrix}0&0\\0&0\end{bmatrix}}, \lambda_3 = \frac{\vartheta^2\begin{bmatrix}0&0\\0&1\end{bmatrix}\vartheta^2\begin{bmatrix}1&0\\0&1\end{bmatrix}}{\vartheta^2\begin{bmatrix}1&0\\0&0\end{bmatrix}\vartheta^2\begin{bmatrix}0&0\\0&0\end{bmatrix}}. \quad (3.1)$$

We have the following numerical evidence of the above representation. For $(\lambda_{10}, \lambda_{20}, \lambda_{30}) = (-0.6, -0.3, 0.6)$, it holds an approximate calcuation

$$\Omega_0 = \begin{pmatrix}0. + 0.9976649279771185i & 0. + 0.40565698917220006i\\ 0. + 0.4056569887269773i & 0. + 1.2660611766736107i\end{pmatrix}.$$

By (3.1), we have

$$-0.6 + 3.9993 * 10^{-33}i, -0.3 + 1.88524 * 10^{-33}i, 0.6 + 3.16375 * 10^{-34},$$

here we used the trancation

$$\overline{\vartheta}\begin{bmatrix}a_1 & a_2\\ b_1 & b_2\end{bmatrix}(\Omega) = \sum_{n=(n_1,n_2)\in\boldsymbol{Z}^2, |n_1|\le 10, |n_2|\le 10}\exp[\pi i(a/2+n)\Omega^t(a/2+n) + 2\pi i(a/2+n)^t b/2].$$

Remark 3.1. *If we start from the other reference curve $C(\lambda)$ with $0 < \lambda_1 < \lambda_2 < \lambda_3 < 1$ equipped with the following homology bisis $\{A_1, B_1, A_2, B_2\}$ as in Fig. 3.2 , we obtain the classical formula found in the work of Igusa [I4]*

$$\lambda_1 = \frac{\vartheta^2\begin{bmatrix}1&0\\0&0\end{bmatrix}\vartheta^2\begin{bmatrix}1&0\\0&1\end{bmatrix}}{\vartheta^2\begin{bmatrix}0&0\\0&0\end{bmatrix}\vartheta^2\begin{bmatrix}0&0\\0&1\end{bmatrix}}, \lambda_2 = \frac{\vartheta^2\begin{bmatrix}1&1\\0&0\end{bmatrix}\vartheta^2\begin{bmatrix}1&0\\0&1\end{bmatrix}}{\vartheta^2\begin{bmatrix}0&1\\0&0\end{bmatrix}\vartheta^2\begin{bmatrix}0&0\\0&1\end{bmatrix}}, \lambda_3 = \frac{\vartheta^2\begin{bmatrix}1&0\\0&0\end{bmatrix}\vartheta^2\begin{bmatrix}1&1\\0&0\end{bmatrix}}{\vartheta^2\begin{bmatrix}0&0\\0&0\end{bmatrix}\vartheta^2\begin{bmatrix}0&1\\0&0\end{bmatrix}}.$$

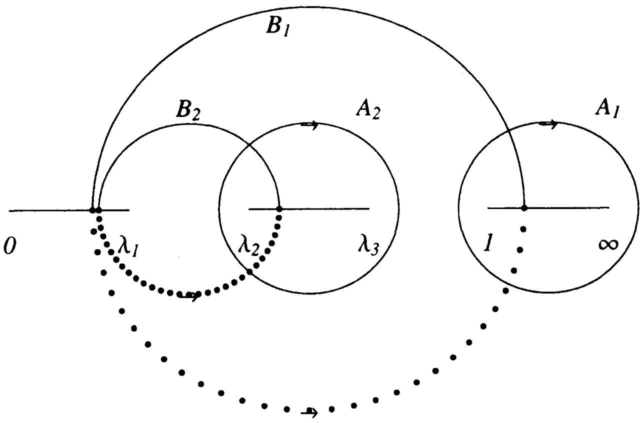

Fig.3.2: Rosenhein cycles

We have a numerical evidence of this formula for the case $C(\lambda)$ with $\lambda = (\lambda_1, \lambda_2, \lambda_3) = (1/4, 1/2, 3/4)$. The approximate period matrix $(\Omega_1, \Omega_2) = \left(\int_{A_j}\omega_i\right), \left(\int_{B_j}\omega_i\right)$ is given by

$$\begin{pmatrix}3.6882 & -10.9961\\ 5.73182 & -6.80204\end{pmatrix}, \begin{pmatrix}0. - 3.6882i & 0. - 10.9961i\\ 0. + 2.04361i & 0. - 4.19408i\end{pmatrix}.$$

So we have the normalized period matrix

$$\Omega_0 = \Omega_1^{-1}\Omega_2 = \begin{pmatrix}1.25352i & 0.755852i\\ 0.755852i & 0. + 1.25352i\end{pmatrix}.$$

By substituting Ω_0 in the formula, we have the approximate values

$$\lambda_1 = 0.25 - 4.13826 * 10^{-34}i, \lambda_2 = 0.5 - 8.27652 * 10^{-34}i, \lambda_3 = 0.75 + 0i,$$

here we used the same trancation of the theta constants.

3.2 Quartic Kummer surface by Kumar

We use (a, b, c) instead of $(\lambda_1, \lambda_2, \lambda_3)$. Consider a curve C of genus 2

$$C = C(a, b, c) : y^2 = x(x - 1)(x - a)(x - b)(x - c).$$

Corresponding quartic Kummer surface is given by

$$\Sigma(a, b, c) : K_2 z_4^2 + K_1 z_4 + K_0 = 0, \tag{3.2}$$

with

$$
\begin{aligned}
K_2 &= z_2^2 - 4z_1z_3 \\
K_1 &= (-2z_2 + 4(a+b+c+1)z_1)z_3^2 \\
&\quad +(-2(bc+ac+c+ab+b+a)z_1z_2 + 4(abc+bc+ac+ab)z_1^2)z_3 - 2abcz_1^2z_2, \\
K_0 &= z_3^4 - 2(bc+ac+c+ab+b+a)z_1z_3^3 + 4(abc+bc+ac+ab)z_1z_2 \\
&\quad +(a^2+b^2+c^2-2ab(a+b+1)-2bc(b+c+1)-2ac(a+c+1) \\
&\quad +a^2b^2+b^2c^2+a^2c^2-2abs(a+b+c+4))z_1^2)z_3^2 \\
&\quad +(-4abcz_1z_2^2 + 4abc(c+a+b+1)z_1^2)z_32 \\
&\quad -2abc(bc+ac+c+ab+b+a)z_1^3)z_3 \\
&\quad +a^2b^2c^2z_1^4.
\end{aligned}
$$

$\Sigma(a,b,c)$ contains 16 ordinary double points (nodes). There are 16 $\boldsymbol{P}^1$'s each of them is coming as the intersection with a tangent (tropes).
List of nodes

$$
\begin{aligned}
&n_1 = (0:0:0:1), n_2 = (0:1:0:0), n_3 = (0:1:1:1), n_4 = (0:1:a:a^2), \\
&n_5 = (0:1:b:b^2), n_6 = (0:1:c:c^2), n_{12} = (1:1:0:abc), n_{13} = (1:a:0:bc), \\
&n_{14} = (1:b:0:ca), n_{15} = (1:c:0:ab), n_{23} = (1:a+1:a:a(b+c)), \\
&n_{24} = (1:b+1:b:b(c+a)), \\
&n_{25} = (1:c+1:c:c(a+b)), n_{34} = (1, a+b, ab, ab(c+1)), \\
&n_{35} = (1, a+c, ca, ac(b+1)), n_{45} = (1:b+c:bc:bc(a+1))
\end{aligned}
$$

List of tropes

$$
\begin{aligned}
&T_0 = (1:0:0:0), T_1 = (0:0:1:0), T_2 = (1:-1:1:0), T_3 = (a^2:-a:1:0), \\
&T_4 = (b^2:-b:1:0), T_5 = (c^2:-c:1:0), T_{12} = (-abc:0:-1:1), T_{13} = (-bc:0:-a:1), \\
&T_{14} = (-ca:0:-b:1), T_{15} = (-ab:0:-c:1), \\
&T_{23} = (-a(b+c):a:-(a+1):1), T_{24} = (-b(c+a):b:-(b+1):1), \\
&T_{25} = (-c(a+b):c:-(c+1):1), T_{34} = (-ab(c+1):ab:-(a+b):1), \\
&T_{35} = (-ca(b+1):ca:-(c+a):1), T_{45} = (-bc(a+1):bc:-(b+c):1).
\end{aligned}
$$

Here, the notation T_3 means $T_3 : a^2z_1 - az_2 + z_3 = 0$ and so on.

4 Some elliptic fibrations according to Kumar

We use Kumar's fibration in [K1] and [K2].

4.1 Kumar's first fibration $\mathcal{R}_1$:

$$
S_{K1} : \eta^2 = 4t(\xi - t - 1)(a\xi - t - a^2)(b\xi - t - b^2)(c\xi - t - c^2) \tag{4.1}
$$

$$
\begin{cases}
t = z_3/z_1 \\
\xi = z_2/z_1 \\
\eta = (z_4/z_1)(\xi^2 - 4t) - \xi(t^2 + (a+b+c+ab+bc+ca)t + abc) \\
\qquad +2t((a+b+c+1)t + (ab+bc+ca+abc)).
\end{cases}
$$

4.2 The 18th fibration $\mathcal{R}_{18}$:

$$\begin{aligned} S_{K18}(a,b,c): y^2 = x^3 &+ ((-4(2abc^2 - bc^2 - ac^2 - ab^2c + 2b^2c - a^2bc - bc - a^2c + 2ac \\ &-ab^2 + 2a^2b - ab)t^2 + 8(bc + ac - c - ab + b - a)t - 8)x^2 \\ &-16(bt - at - 1)(act - at - 1)(act - ct - abt + bt - 1)(bct - abt - 1)(bct - ct - 1)x. \end{aligned} \tag{4.2}$$

Elliptic fibration $\pi : (t,x,y) \mapsto t$. The singular composition is given by $I_4 + 5I_2 + I_1 + III^*$ with a two-torsion section $\{x = y = 0\}$.

$\pi^{-1}(0)$	$\pi^{-1}(t_1)$	$\pi^{-1}(t_2)$	$\pi^{-1}(t_3)$	$\pi^{-1}(t_4)$	$\pi^{-1}(t_5)$	$\pi^{-1}(t_0)$	$\pi^{-1}(\infty)$
I_4	I_2	I_2	I_2	I_2	I_2	I_1	III^*
A_3	A_1	A_1	A_1	A_1	A_1	-	E_7

,

with

$$\begin{aligned} t_0 = (&-a^2b^2 + 2a^2b^3 - a^2b^4 + 2a^3bc + 2ab^2c - 4a^2b^2c - 2ab^3c + 2a^2b^3c \\ &-2a^3b^3c + 2a^2b^4c - a^4c^2 - 4a^3bc^2 + 2a^4bc^2 - b^2c^2 + 2ab^2c^2 + 4a^2b^2c^2 \\ &+2a^3b^2c^2 - a^4b^2c^2 - 4a^2b^3c^2 + 2a^3b^3c^2 - a^2b^4c^2 + 2a^3c^3 - 2abc^3 + 2a^2bc^3 \\ &-2a^3bc^3 + 2b^2c^3 - 4ab^2c^3 + 2ab^3c^3 - a^2c^4 + 2abc^4 - b^2c^4) \\ &/(4abc(a-1)(b-1)(c-1)(a-b)(b-c)(c-a)), \end{aligned} \tag{4.3}$$

and

$$t_1 = \frac{1}{b-a}, t_2 = \frac{1}{b(c-a)}, , t_3 = \frac{1}{(a-1)(c-b))}, t_4 = \frac{1}{a(c-1)}, t_5 = \frac{1}{c(b-1)}.$$

Proposition 4.1. *$S_{K18} \in \mathcal{F}_N$ if and only if $t_0 = 0$.*

Remark 4.1. *Hashimoto-Murabayashi [H-M] (Theorem 2.9 p. 285) showed the condition $C_{HM}(a,b,c) = 0$ that the Kummer surface S_{K1} is coming from an abelian variety with $\sqrt{5}$ action, where*

$$\begin{aligned} C_{HM}(a,b,c) = 4(a^2bc - ab^2c)(b - b^2 - c + a^2c + (1-a)c^2) - (a(b-c) \\ +a^2(1+b)c + (1-a)bc^2 - b^2(a+c))^2. \end{aligned}$$

In fact, it holds $C_{HM}(a,b,c) =$ the numerator of t_0 up to a rotation of parameters.

4.3 The 23rd fibration $\mathcal{R}_{23}$:

$$y^2 = x^3 - 2(t^3 - \frac{I_4}{12}t + \frac{I_2I_4 - 3I_6}{108})x^2 + ((t^3 - \frac{I_4}{12}t + \frac{I_2I_4 - 3I_6}{108})^2 + I_{10}(t - \frac{I_2}{24}))x. \tag{4.4}$$

The singular composition is given by $I_5^* + 6I_2 + I_1$, where I_2, I_4, I_6, I_{10} are Igusa-Clebsch invariants those are described as symmetric polynomials in a, b, c.

5 Construction of an explicit period vector

In this section we give an explicit construction of the period map for the family $\mathcal{F}_{L_0}$. For the moment we fix a reference surface S_R that is a member of the family $\mathcal{F}_{L_0} = \{S(\alpha,\beta,\gamma,\delta)\}$. We put $(\alpha,\beta,\gamma,\delta) = (4,1,5,18)$. Then we get

$$S_R : y^2 = x^3 + (-12t^4 - 5t^5)x + t^5 - 2t^6 + 18t^7. \tag{5.1}$$

Set $F(x,t) = x^3 + (-12t^4 - 5t^5)x + t^5 - 2t^6 + 18t^7$. The discriminant $\Delta(t)$ of $F(x,t)$ with respect to x is given by

$$\Delta(t) = -t^{10}(-27 + 108t + 5832t^2 + 10584t^3 - 5148t^4 + 500t^5).$$

The roots of $\Delta(t) = 0$ are given by real simple ones

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (-0.422264, -0.0862632, 0.0569883, 3.68146, 7.06608)$$

together with $\alpha_0 = 0$ that is a root of multiplicity 10. So we have a singular fiber of type II^* at $t = 0$ and five singular fibers of type I_1 at $t = \alpha_i (i = 1, \ldots, 5)$. At $t = \infty$ we have a singular fiber of type III^*. The sublattice L_0 is realized by the components of $\pi^{-1}(0)$ and $\pi^{-1}(\infty)$ together with the section s_0 and the general fiber f.

As a first step, we make a table of local monodromies of a fixed generic fiber $E_0 = \pi^{-1}(\sqrt{-1})$ of the elliptic surface $(S_R, \pi, \boldsymbol{P}^1)$ equipped with a projection $\pi : (x, y, t) \mapsto t$.

The elliptic curve at the base point $b = \sqrt{-1}$ is given by

$$E_0 = \pi^{-1}(i) : y^2 = x^3 - (12 + 5\sqrt{-1})x + (2 - 17\sqrt{-1}).$$

As a double cover of the x-sphere, it has four ramification points

$$(br_1, br_2, br_3, br_\infty) = (-3.5396 - 0.0272802i, -0.244328 - 1.19164i, 3.78392 + 1.21892i, \infty).$$

We make two 1-cycles γ_1, (γ_2 resp.) that projection goes around br_1 and br_2 in the negative sense (br_2 and br_3 in the negative sense, resp.) so that we have the intersection $\gamma_1 \cdot \gamma_2 = 1$ (see Fig. 5.1).

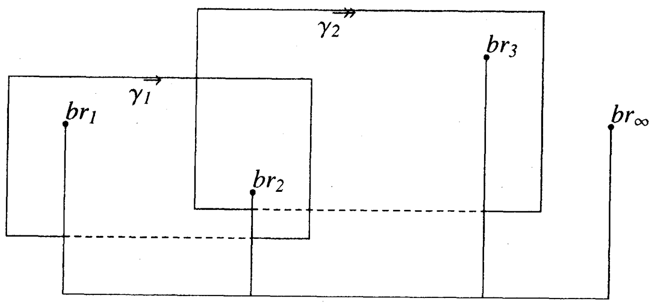

Fig.5.1: Basis of $H_1(\pi^{-1}(i), \boldsymbol{Z})$

Let δ_i ($i = 0, 1, 2, 3, 4, 5, \infty$) be a closed oriented arc on the t plane starting at $b = \sqrt{-1}$ and going around $t = \alpha_i$ in the positive sense (see Fig. 5.2). The loop δ_i induces a monodromy transformation of the system $\{\gamma_1, \gamma_2\}$. Let us denote it by M_i as a left action. We call them local monodromies.

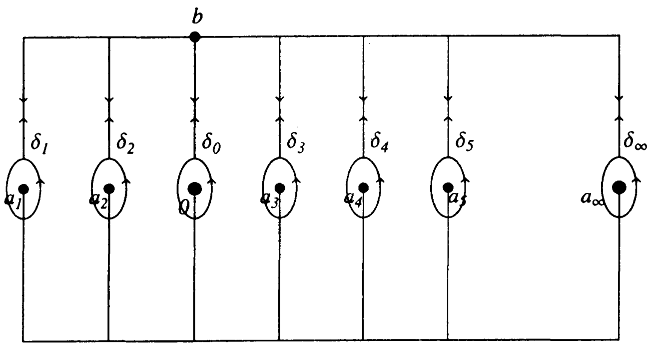

Fig.5.2: Singular fibers of S_R and δ_i

Proposition 5.1. *The local monodromies are given by the following table:*

t	α_1	α_2	α_0	α_3	α_4	α_5	α_∞
M_i	$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$
type	I_1	I_1	II^*	I_1	I_1	I_1	III^*
inv. cycle	γ_2	γ_2		γ_1	γ_1	γ_1	

Table 5.1

Note that for the singular fiber of type I_1, it appears a cycle on E_0 that is invariant under the local monodromy which is indicated in the Table 8.1.

Next we construct a basis $\{\Gamma_1^*, \ldots, \Gamma_5^*\}$ of the transcendental lattice $Tr(S_R)$. Let δ be an oriented arc starting from the base point $\sqrt{-1}$ on the t-plane and set $j \in \{1, 2\}$. We make a 2-chain $\Gamma(\delta, j)$ obtained by the continuation of γ_j along δ. We define the orientation of $\Gamma(\delta, j)$ by the ordered pair of those of δ and γ_j. If δ is a loop returning back to the starting cycle γ_j, it becomes to be a 2-cycle on S_R. According to this notation we define the following 2-cycles on S_R (see Fig. 5.3):

$$G_1^* = \Gamma(\delta_1\delta_2^{-1}, \gamma_1), G_2^* = \Gamma(\delta_2\delta_0, \gamma_1), G_3^* = \Gamma(\delta_0\delta_3, \gamma_2)$$
$$G_4^* = \Gamma(\delta_3\delta_4^{-1}, \gamma_2), G_5^* = \Gamma(\delta_4\delta_5^{-1}, \gamma_2).$$

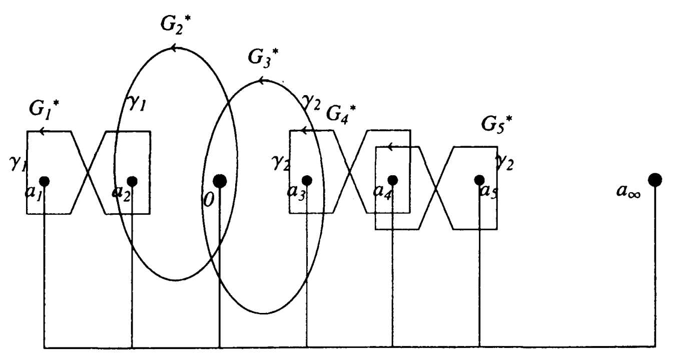

Fig.5.3: Basis of $Tr(S_R)$

By a direct calculation we obtain

Proposition 5.2. *The intersection matrix of the system $\{G_1^* \dots, G_5^*\}$ is given by*

$$\check{B} = \begin{pmatrix} -2 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

Set

$$T_n = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

By a base change ${}^t(\Gamma_1^*, \dots, \Gamma_5^*) = T_n{}^t(G_1^* \dots, G_5^*)$, the intersection matrix of the system $\{\Gamma_1^*, \dots, \Gamma_5^*\}$ becomes to be $B_0 = \langle -2 \rangle \oplus U \oplus U$, the expected one. So, $\{\Gamma_1^*, \dots, \Gamma_5^*\}$ is a system of generators of $Tr(S_R)$, the generic transcendental lattice.

Let $\ell_i (i = 1, 2, 3, 4, 5)$ be an oriented line segment in the upper half plane starting from $t = \infty$ terminating at br_i. We make another system of 2-cycles on S_R:

$$Cc_1 = \Gamma(\ell_1, \gamma_2), Cc_2 = \Gamma(\ell_2, \gamma_2)$$
$$Cc_3 = \Gamma(\ell_3, \gamma_1), Cc_4 = \Gamma(\ell_4, \gamma_1), Cc_5 = \Gamma(\ell_5, \gamma_1).$$

We have the following intersection matrix $M_{gc} = (G_i^* \cdot Cc_j)_{1 \le i,j \le 5}$:

$$M_{gc} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Set

$$Pcc = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

We define a system

$${}^t(G_1, G_2, G_3, G_4, G_5) = Pcc^t(Cc_1, Cc_2, Cc_3, Cc_4, Cc_5). \tag{5.2}$$

By an easy matrix calculation, we obtain.

Proposition 5.3. *It holds*

$$G_i^* \cdot G_j = \delta_{ij} \ (1 \le i, j \le 5).$$

Recall that we defined

$$^t(\Gamma_1^*, \dots, \Gamma_5^*) = T_n{}^t(G_1^* \dots, G_5^*).$$

Setting $^t(\Gamma_1, \dots, \Gamma_5) = T_q{}^t(G_1^* \dots, G_5^*)$, it holds

$$Tq = {}^tT_n^{-1t}\check{B}^{-1} = \begin{pmatrix} 0 & 0 & -(1/2) & 0 & -(1/2) \\ 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Proposition 5.4. *The systems $\{\Gamma_1, \dots, \Gamma_5\}$ and $\{\Gamma_1^*, \dots, \Gamma_5^*\}$ are mutually dual systems in the sense $\Gamma_i \cdot \Gamma_j^* = \delta_{ij}$ $(1 \le i, j \le 5)$.*

We have an alternative expression of the system $\{G_1^*, \dots, G_5^*\}$. Recall that we denoted the generic Picard lattice of $S(\alpha, \beta, \gamma, \delta)$ by L_0.

Proposition 5.5. *It holds the following equalities modulo L_0(see Fig. 5. 4).*

$$G_1^* \equiv -\Gamma[[cr_1, cr_2], \gamma_2], G_2^* \equiv -\Gamma[[cr_2, 0], \gamma_2], G_3^* \equiv -\Gamma[[0, , cr_3], \gamma_1],$$
$$G_4^* \equiv \Gamma[[cr_3, , cr_4], \gamma_1], G_5^* \equiv \Gamma[[cr_4, , cr_5], \gamma_1].$$

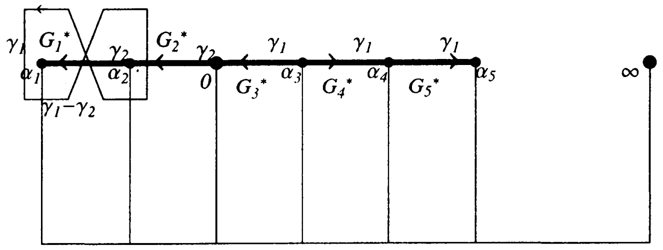

Fig. 5.4: Transformed basis of $Tr(S_R)$

Put

$$\eta_i = \int_{\Gamma_i} \omega \ (i = 1, \dots, 5), \ \eta = (\eta_1, \dots, \eta_5).$$

The Riemann-Hodge period relation is given by

$$\eta B_0{}^t\eta = 0, \ \eta B_0{}^t\overline{\eta} > 0.$$

By putting

$$\check{\eta_i}^* = \int_{G_i^*} \varphi \ (i = 1, \dots, 5).$$

These relations are translated to the relation

$$(\check{\eta_1}^*, \dots, \check{\eta_5}^*)\check{B}^{-1t}(\check{\eta_1}^*, \dots, \check{\eta_5}^*) = 0, (\check{\eta_1}^*, \dots, \check{\eta_5}^*)\check{B}^{-1t}(\overline{\check{\eta_1}^*}, \dots, \overline{\check{\eta_5}^*}) > 0.$$

Remark 5.1. (Numerical evidence) *By using MATHEMATICA we can obtain the following approximate values of the period vector of S_R. By making an approximation of the double integrals we obtain*

$$-\frac{1}{2}\int_{\Gamma[\gamma_2,[cr_1,cr_2]]} \varphi = 2.11, -\frac{1}{2}\int_{\Gamma[\gamma_2,[cr_2,0]]} \varphi = -7.8, -\frac{1}{2}\int_{\Gamma[\gamma_1,[0,cr_3]]} \varphi = 7.1i,$$
$$\frac{1}{2}\int_{\Gamma[\gamma_1,[cr_3,cr_4]]} \varphi = -5.16i, \frac{1}{2}\int_{\Gamma[\gamma_1,[cr_4,cr_5]]} \varphi = -0.628i.$$

According to Prop. 5.5 we have the period vector

$$(\check{\eta_1}^*, \check{\eta_2}^*, \check{\eta_3}^*, \check{\eta_4}^*, \check{\eta_5}^*) = 2(2.11, -7.8, 7.1i, -5.16i, -0.63i)$$

for our reference surface $S_R = S(4, 1, 5, 18)$. We see the period relation is approximately satisfied.

6 Modular map

For a curve of genus two:

$$y^2 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3), \tag{6.1}$$

the Igusa-Clebsch invariants are given by

Proposition 6.1.

$$\begin{cases} I_2 = 2(3s_1^2 - 2(s_2+4s_3)s_1 + 3s_2^2 - 8s_2 + 12s_3), \\ I_4 = 4(-3s_3s_1^3 + (s_2^2 - s_3s_2 + s_3^2 + 3s_3)s_1^2 \\ \qquad +(-s_2^2 + 11s_3s_2 - 3s_3)s_1 - 3s_2^3 + (3s_3+1)s_2^2 - 3s_3^2s_2 - 18s_3^2), \\ I_6 = -24s_2^3 + 48s_2^4 + 24s_1^4s_3 + 104s_2^2s_3 + 53s_2^3s_3 - 36s_3^2 + 168s_2s_3^2 + 199s_2^2s_3^2 - 180s_3^3 \\ \qquad -42s_2s_3^3 - 36s_3^4 + s_1^3(-8s_2^2 - 24s_3 + 307s_2s_3 - 73s_3^2) \\ \qquad +s_1^2(8s_2^2 - 36s_2^3 + 123s_2s_3 + 450s_2^2s_3 - 53s_3^2 + 396s_2s_3^2 + 72s_3^3) \\ \qquad +s_1(20s_2^3 + 76s_2s_3 + 328s_2^2s_3 + 189s_2^3s_3 - 168s_3^2 + 826s_2s_3^2 + 189s_2^2s_3^2 + 294s_3^3) \\ I_{10} = \lambda_1^2\lambda_2^2\lambda_3^2(\lambda_1-1)^2(\lambda_2-1)^2(\lambda_3-1)^2(\lambda_1-\lambda_2)^2(\lambda_2-\lambda_3)^2(\lambda_3-\lambda_1)^2, \end{cases} \tag{6.2}$$

where $s_1 = \lambda_1 + \lambda_2 + \lambda_3$, $s_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$, $s_3 = \lambda_1\lambda_2\lambda_3$.

By comparing (2.3) and (4.4) we obtain

Theorem 6.1. *It holds*

$$(\alpha : \beta : \gamma : \delta) = (\frac{1}{9}I_4 : \frac{1}{27}(-I_2I_4 + 3I_6) : 8I_{10} : \frac{2}{3}I_2I_{10}). \tag{6.3}$$

Proposition 6.2. *We have an alternative representation of S_{K18}:*

$$\begin{cases} z^2 = x^3 + p_0(t)x^2 + q_0(t)x, \\ \text{with} \\ p_0(t) = a_0 + a_1t + a_2t^2 \\ q_0(t) = \frac{1}{4}a_0^2 + \frac{1}{2}a_1a_0t + \frac{1}{4}(a_1^2 + 2a_2a_0)t^2 + \frac{1}{2}a_1a_2t^3 + b_4t^4 + b_5t^5. \end{cases} \tag{6.4}$$

It has a $\sqrt{5}$ action if and only if $4b_4 = a_2^2$.

By comparing (2.5) and (6.4)we obtain

Theorem 6.2. *The affine parameter $X = \frac{\mathcal{B}}{\mathcal{A}^3}, Y = \frac{\mathcal{C}}{\mathcal{A}^5}$ have the expression*

$$\begin{cases} X = -50\frac{a_1}{a_2^3} \\ Y = 2^4 5^5 \frac{a_0}{a_2^5}, \end{cases} \tag{6.5}$$

where

$$\begin{cases} a_0 = -8, \\ a_1 = -8(a - b + ab + c - ac - bc), \\ a_2 = 4(ab - 2a^2b + ab^2 - 2ac + a^2c + bc + a^2bc - 2b^2c + ab^2c + ac^2 + bc^2 - 2abc^2). \end{cases} \tag{6.6}$$

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